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SOME UNIFIED INTEGRALS ASSOCIATED WITH THE GENERALIZED STRUVE FUNCTION

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Abstract. This paper is devoted for the study of a new generalization of Struve type function. In this paper, we establish four new integral formulas involving the Galué type Struve function, which are express in term of the generalized (Wright) hypergeometric functions. The result established here are general in nature and are likely to find useful in applied problem of sciences, engineering and technology.

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1. INTRODUCTION

Recently, Nisar *et al.* [15], defined as following generalized form of Struve function named as generalized Galué type Struve function (GTSF):

$$
(1) \qquad {}_a w_{p,\,b,\,c,\,\xi}^{\lambda,\,\mu}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma(\lambda k+\mu)\,\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)} \left(\frac{z}{2}\right)^{2k+p+1}
$$

$$
(a \in \mathbb{N},\, p,\, b,\, c \in \mathbb{C}),
$$

where $\lambda > 0$, $\xi > 0$ and μ is an arbitrary parameter. For the more details on the Struve function and its generalizations, one may refer to the recent papers [5], [6], [8], [19], [20] and [21].

Particularly, when $\lambda = a = 1$, $\mu = 3/2$ and $\xi = 1$ in equation (1), it reduces to generalization of Struve function which is defined by [18], as under:

$$
(2) \qquad H_{p,\,b,\,c}(z) = \sum_{k=0}^{\infty} \frac{(-c)^k}{\Gamma\left(k+\frac{3}{2}\right)\,\Gamma\left(k+p+\frac{b+2}{2}\right)} \left(\frac{z}{2}\right)^{2k+p+1}
$$

$$
(p,\, b,\, c \in \mathbb{C}).
$$

Further, the detailed study related to the function $H_{p,\,b,\,c}(z)$ and its particular cases can be seen in [1], [2], [12], [13], [14] and [16].

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For our present investigation, we need to recall the following Oberhettinger's integral formula [17]:

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} dx = 2\lambda a^{-\lambda}\left(\frac{a}{2}\right)^\mu \frac{\Gamma(2\mu)\Gamma(\lambda-\mu)}{\Gamma(1+\lambda+\mu)}, \tag{3}$$

provided $0 < \Re(\mu) < \Re(\lambda)$.

We also recall the following Lavoie-Trottier integral formula [9]:

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} dx = \left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{4}$$

provided $\Re(\alpha) > 0, \Re(\beta) > 0$. In recent years, various useful integral formulas associated with the variety of special functions have been studied by several authors, see [3],[4],[10] and [11].

The generalized Wright hypergeometric function ${}_p\psi_q(z)$ [24] (see, for detail, Srivastava and Karlsson [23]), for $z \in \mathbb{C}$ complex, $a_i,\ b_j \in \mathbb{C}$ and $\alpha_i,\ \beta_j \in \Re$, where $(\alpha_i,\ \beta_j \neq 0\,;\, i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$, is defined as under:

$$ {}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c} (a_i,\ \alpha_i)_{\ 1,p} \\ (b_j,\beta_j)_{\ 1,q} \end{array} \Big| z\right] = \sum_{k=0}^\infty \frac{\prod_{i=1}^p \Gamma(a_i+\ \alpha_i k)\, z^k}{\prod_{j=1}^q \Gamma(b_j+\ \beta_j k)\, k\,!}, \tag{5}$$

under the condition:

$$\sum_{j=1}^q \beta_j - \sum_{i=1}^p \alpha_i > -1. \tag{6}$$

It is noted that the generalized (Wright) hypergeometric function ${}_p\psi_q$ in (5) whose asymptotic expansion was investigated by Fox [7] and Wright is an interesting further generalization of the generalized hypergeometric series as follow:

$$ {}_p\Psi_q\left[\begin{array}{c} (a_1,\, 1)\,,\ \ldots,\ (a_p,\, 1) \\ (b_1,\, 1)\,,\ \ldots,\ (b_q,\, 1) \end{array}\ z\right] = \frac{\prod_{j=1}^p \Gamma\left(\alpha_j\right)}{\prod_{j=1}^q \Gamma\left(\beta_j\right)}\, {}_pF_q\left[\begin{array}{c} \alpha_1,\ \ldots, \alpha_p\,; \\ \beta_1,\ \ldots, \beta_q\,; \end{array}\ z\right], \tag{7}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [20, Section 1.5])

$$ \begin{aligned} {}_pF_q\left[\begin{array}{c} \alpha_1,\ \ldots, \alpha_p\,; \\ \beta_1,\ \ldots, \beta_q\,; \end{array}\ z\right] &= \sum_{n=0}^\infty \frac{(\alpha_1)_n\cdots(\alpha_p)_n z^n}{(\beta_1)_n\cdots(\beta_q)_n n!} \\ &=_p F_q(\alpha_1,\ \ldots, \alpha_p; \beta_1,\ \ldots, \beta_q; z), \end{aligned} \tag{8}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [22]):

$$ \begin{aligned} (\lambda)_n &= \left\{ \begin{array}{l} 1\,(n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) \quad (n \in \mathbb{N} = \{1, 2, 3,\ \ldots\}) \end{array}\right. \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C}/\mathbb{Z}_0), \end{aligned} \tag{9}$$

and $\mathbb{Z}_0$ denotes the set of nonpositive integers.

The main object of this paper is to establish certain new integrals involving the Galué type Struve function, which are express in term of the generalized (Wright) hypergeometric functions. In order to obtain our main results, we use the generalized Galué type Struve function (1), with suitable arguments, as the integrand of the integrals (3) and (4).

2. Main Results

We will now give our main integral formulas as under:

Theorem 2.1. *Let* $a \in \mathbb{N}$ λ, p, b, $c \in \mathbb{C}$; $\upsilon > 0$ *and* δ *is an arbitrary parameter be such that* $0 < \Re(\mu) < \Re(\lambda+p+1)$ *then there hold the following results:*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} {}_a w_{p,b,c,\xi}^{\upsilon,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx \tag{10}$$

$$= 2^{-\mu-p}a^{\mu-\lambda-p-1}y^{p+1}\Gamma(2\mu)$$

$$\times {}_3\psi_4\left[\begin{matrix}(\lambda+p+2,\,2),(\lambda-\mu+p+1,\,2),(1,1)\,; \\ (\delta,\upsilon),\left(\frac{p}{\xi}+\frac{b}{2}+1,a\right),(\lambda+p+1,2),(\lambda+\mu+p+2,2)\end{matrix}\;\frac{-cy^2}{4a^2}\right].$$

Proof. By making use of (1) in the integrand of (10), and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} {}_a w_{p,\,b,\,c,\,\xi}^{\upsilon,\delta}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx \tag{11}$$

$$=\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma(\upsilon k+\delta)\,\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)}\left(\frac{y}{2}\right)^{2k+p+1}$$

$$\times\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-(\lambda+2k+p+1)}dx.$$

Now, on applying the integral formula (3) to the above integral in right hand side of (11), and obtain the following expression:

$$=\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma(\upsilon k+\delta)\,\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)}\left(\frac{y}{2}\right)^{2k+p+1}$$

$$\times\left(\frac{a}{2}\right)^{\mu}2\,(\lambda+p+1+2k)\,a^{-(\lambda+p+1+2k)}\frac{\Gamma(2\mu)\,\Gamma(\lambda-\mu+p+1+2k)}{\Gamma(\lambda+\mu+p+2+2k)},$$

$$=2^{-\mu-p}a^{\mu-\lambda-p-1}y^{p+1}\Gamma(2\mu)\sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma(\upsilon k+\delta)}$$

$$\times\frac{\Gamma(\lambda+p+2+2k)\,\Gamma(\lambda-\mu+p+1+2k)}{\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)\Gamma(\lambda+p+1+2k)\,\Gamma(\lambda+\mu+p+2+2k)}\left(\frac{y}{2a}\right)^{2k},$$

which in accordance with the definition (5), yield to the result (10). This completes the proof of the theorem. □

Theorem 2.2. *Suppose $a \in \mathbb{N}$ λ, p, b, $c \in \mathbb{C}$; $\upsilon > 0$ and δ is an arbitrary parameter be such that $0 < \Re(\mu) < \Re(\lambda + p + 1)$, then*

$$\int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} {}_a w^{\upsilon,\delta}_{p,\,b,\,c,\,\xi}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx \tag{12}$$
$$= 2^{-\mu-2p}a^{\mu-\lambda-1}y^{p+1}\Gamma\left(\lambda-\mu+1\right)$$
$$\times {}_3\psi_4\left[\begin{array}{c} (\lambda+p+2,\ 2),(2\mu+2p,4),(1,1)\ ; \\ (\delta,\upsilon),\left(\frac{p}{\xi}+\frac{b}{2}+1,a\right),(\lambda+p+1,2),(\lambda+\mu+2p+2,4)\ ; \end{array} \frac{-cy^2}{4}\right].$$

Proof. By similar manner as in proof of Theorem 2.1, one can easily prove the integral formula (12). Therefore, we omit the detailed proof. □

Theorem 2.3. *The following integral formulas holds:*

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} \tag{13}$$
$$\times {}_a w^{\upsilon,\delta}_{p,\,b,\,c,\,\xi}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$= \left(\frac{2}{3}\right)^{2\alpha}\left(\frac{y}{2}\right)^{p+1}\Gamma\left(2\alpha\right)$$
$$\times {}_2\psi_3\left[\begin{array}{c} (\beta+p+1,2),(1,1)\,; \\ (\delta,\upsilon),\left(\frac{p}{\xi}+\frac{b}{2}+1,a\right),(2\alpha+\beta+p+1,2)\,; \end{array} \frac{-cy^2}{4}\right],$$

provided $a \in \mathbb{N}$ λ, p, b, $c \in \mathbb{C}$; $\upsilon > 0$ and δ an arbitrary parameter, such that $\Re(\alpha) > 0, \Re\left(\beta+p+1\right) > 0$.

Proof. By making use of (1) in the ingrand of (13) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} \tag{14}$$
$$\times {}_a w^{\upsilon,\delta}_{p,\,b,\,c,\,\xi}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx$$
$$= \sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma\left(\upsilon\, k+\delta\right)\,\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)}\left(\frac{y}{2}\right)^{2k+p+1}$$
$$\times\int_0^1 x^{\alpha-1}(1-x)^{2(\beta+p+1+2k)-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta+p+1+2k-1}dx,$$

we can apply the integral formula (4) to the integral in (14) and obtain the following expression:

$$= \sum_{k=0}^{\infty}\frac{(-c)^k}{\Gamma\left(\upsilon k+\delta\right)\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)}\left(\frac{y}{2}\right)^{2k+p+1}$$
$$\times\left(\frac{2}{3}\right)^{2\alpha}\frac{\Gamma\left(2\alpha\right)\Gamma\left(\beta+p+1+2k\right)}{\Gamma\left(2\alpha+\beta+p+1+2k\right)},$$

$$= \left(\frac{2}{3}\right)^{2\alpha} \left(\frac{y}{2}\right)^{p+1} \Gamma(2\alpha)$$

$$\times \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(\beta+p+1+2k)}{\Gamma(\upsilon k+\delta)\,\Gamma\left(ak+\frac{p}{\xi}+\frac{b+2}{2}\right)\Gamma(2\alpha+\beta+p+1+2k)} \left(\frac{y}{2}\right)^{2k},$$

In accordance with the definition of (5), we obatain the result (15). This completes the proof of the theorem. □

Following the similar procedure, we further obtain the integral formula (15) as under:

Theorem 2.4. *For $a \in \mathbb{N}$ λ, p, b, $c \in \mathbb{C}$; $\upsilon > 0$ and δ an arbitrary parameter such that $\Re(\beta) > 0, \Re(\alpha+p+1) > 0$, we have*

$$(15) \quad \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1}$$

$$\times {}_a w^{\upsilon,\delta}_{p,b,c,\xi}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx$$

$$= \left(\frac{2}{3}\right)^{2(\alpha+p+1)} \left(\frac{y}{2}\right)^{p+1} \Gamma(\beta)$$

$$\times {}_2\psi_3 \left[\begin{array}{c} (2\alpha+2p+2,\,4)\,,(1,1)\,; \\ (\delta,\upsilon)\,,\left(\frac{p}{\xi}+\frac{b}{2}+1,a\right),(2\alpha+\beta+2p+2,\,4)\,; \end{array} \frac{-4cy^2}{81} \right].$$

3. Special Cases

In this section, we derive certain new integrals involving known generalized Struve function due to Orhan and Yagmur ([18]), as particular cases of our main results.

To this end, if we set $\upsilon = a = 1, \delta = 3/2$ and $\xi = 1$ in the results of Theorem 2.1 to 2.4, we obtain the following four corollaries associated with the generalized Struve function (2):

Corollary 3.1. *Let the condition of Theorem 2.1 be satisfied, then we have*

$$(16) \quad \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} H_{p,b,c}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{-\mu-p} a^{\mu-\lambda-p-1} y^{p+1} \Gamma(2\mu)$$

$$\times {}_3\psi_4 \left[\begin{array}{c} (\lambda+p+2,2)\,,(\lambda-\mu+p+1,2)\,,(1,1)\,; \\ \left(p+\frac{b+2}{2},1\right),(\lambda+p+1,2)\,,(\lambda+\mu+p+2,2)\,,(3/2,1)\,; \end{array} \frac{-cy^2}{4a^2} \right].$$

Corollary 3.2. *If the condition of Theorem 2.2 be satisfied, then the following integral holds:*

$$(17) \quad \int_0^\infty x^{\mu-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\lambda} H_{p,b,c}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{-\mu-2p} a^{\mu-\lambda-1} y^{p+1} \Gamma(\lambda-\mu+1)$$

$$\times {}_3\psi_4 \left[\begin{array}{c} (\lambda+p+2,2), (2\mu+2p,4), (1,1); \\ \left(p+\frac{b+2}{2},1\right), (\lambda+p+1,2), (\lambda+\mu+2p+2,4), (3/2,1); \end{array} \frac{-cy^2}{4} \right].$$

Corollary 3.3. *Under the valid condition of Theorem 2.3, we have*

$$\begin{aligned} (18) \qquad & \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} \\ & \times H_{p,b,c}\left(y\left(1-\frac{x}{4}\right)(1-x)^2\right)dx \\ & = \left(\frac{2}{3}\right)^{2\alpha}\left(\frac{y}{2}\right)^{p+1}\Gamma(2\alpha) \\ & \times {}_2\psi_3 \left[\begin{array}{c} (\beta+p+1,2), (1,1); \\ \left(p+\frac{b+2}{2},1\right), (2\alpha+\beta+p+1,\,2), (3/2,1); \end{array} \frac{-cy^2}{4} \right]. \end{aligned}$$

Corollary 3.4. *The following integral holds*

$$\begin{aligned} (19) \qquad & \int_0^1 x^{\alpha-1}(1-x)^{2\beta-1}\left(1-\frac{x}{3}\right)^{2\alpha-1}\left(1-\frac{x}{4}\right)^{\beta-1} \\ & \times H_{p,b,c}\left(yx\left(1-\frac{x}{3}\right)^2\right)dx \\ & = \left(\frac{2}{3}\right)^{2(\alpha+p+1)}\left(\frac{y}{2}\right)^{p+1}\Gamma(\beta) \\ & \times {}_2\psi_3 \left[\begin{array}{c} (2\alpha+2p+2,\,4), (1,1); \\ \left(p+\frac{b+2}{2},1\right), (2\alpha+\beta+2p+2,\,4), (3/2,1); \end{array} \frac{-4cy^2}{81} \right], \end{aligned}$$

provided the condition of Theorem 2.4 are satisfied.

Conclusion: Certain unified integral representations for the generalized Struve function and its special cases are derived in this study. In this sequel, one can easily obtain integral representation of more generalized special function, which has much application in physics and engineering science.

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